The maths of mince pies

Eugenia Cheng
School of the Art Institute of Chicago

November 28th, 2016

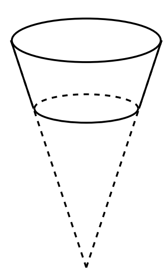
Abstract

The mince pie, aka frustrum, is an interesting mathematical object, as well as being delicious. In this note we study mince pies as a tool for elucidating various techniques of basic mathematics and calculus including volumes, areas, solving quadratic equations, and differentiation.

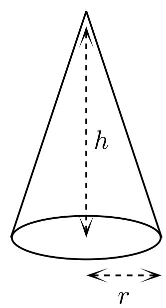
Contents

1 Introduction

A mince pie is essentially a frustrum, that is, a cone with a smaller cone removed from the vertex end.

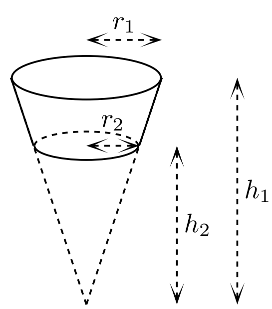


However, the normal way of studying a frustrum is via the two cones, which is not very helpful for mince pies as the cones do not actually exist. The volume of a cone is usually expressed in terms of the radius and height, like this:



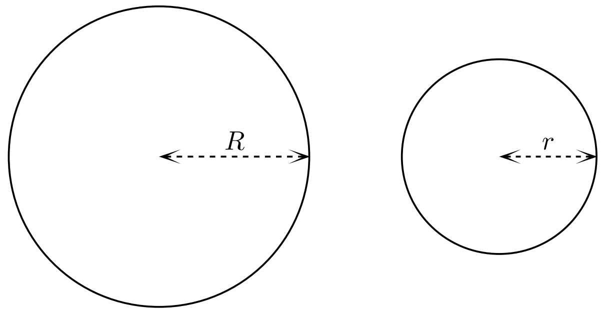
$$V = \frac{\pi}{3} r^2 h$$

So the volume of a frustrum is the volume of the large cone minus the volume of the small cone



$$V_1 - V_2 = \frac{\pi}{3} r_1^2 h_1 - \frac{\pi}{3} r_2^2 h_2$$

This is particularly unhelpful for a mince pie as we are unlikely to know either h_1 or h_2, and even r_1 is not usually a variable we are controlling. We are more likely to make a pie from two circles of pastry, one for the base and sides and one for the top.



So our aim is to express the volume of the frustrum in terms of these variables instead and use this to study various aspects of mince pies.

Assumptions

Of course we make various simplifying assumptions about our mince pies along the way, because we are mathematicians and we live in a dream world.

Let R be the radius of the large circle of pastry and let r be the radius of the small circle for the lid of the mince pie. (It doesn't matter what units we use as long as we are consistent throughout.)

We assume:

1. that the circles and volumes are all perfectly regular,

2. that the angle of the mince pie is $30°$

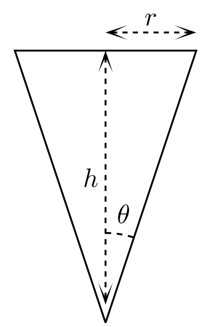
   

   and

3. that the radius of the lid of the mince pie is the same as the radius of the base.

2 Volume

The main task is to derive a formula for the volume of the frustrum depending only on our relevant variables R and r. We start by eliminating h from the volume of a cone.



$$\tan\theta = \frac{r}{h}$$

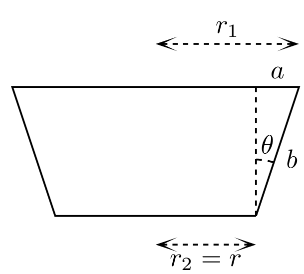
so

$$h = \frac{r}{\tan\theta}.$$

Thus the volume of the cone becomes

$$\begin{aligned} V_{\text{cone}} &= \frac{\pi}{3} r^2 h \\ &= \frac{\pi r^2}{3\tan\theta} \end{aligned}$$

Now observe for our frustrum the cross-section looks like this:



We

$$r = r_2$$

by definition, and

$$R = r + b$$

because of how the large circle of pastry forms the base and sides of the pie. Now

$$\begin{aligned} r_1 &= r + a \\ a &= b\sin\theta \\ \text{so } r_1 &= r + b\sin\theta \\ &= r(1-\sin\theta) + (r+b)\sin\theta \\ &= r(1-\sin\theta) + R\sin\theta \end{aligned}$$

Note that the second step is a trick to get the expression $r + b$ in. The final expression essentially expresses r_1 as a weighted average (mean) of r and R, weighted by $\sin\theta$. This is saying that the top radius of a pie is somewhere between the radius of the large circle of pastry and the small one, and that when the sides are steeper (small angle θ) the top radius is closer to the botter radius (like a cylinder).

At this point we invoke our simplifying assumption $\theta = 30°$. So

$$\begin{aligned}\tan\theta &= \frac{1}{\sqrt{3}}\\ \sin\theta &= \frac{1}{2}\end{aligned}$$

Then

$$\begin{aligned}r_1 &= \frac{R+r}{2}\\ \text{so } V_{\text{cone}} &= \frac{\pi r^3}{3\tan\theta}\\ &= \frac{\sqrt{3}}{3}\pi r^3\\ &= \frac{\pi r^3}{\sqrt{3}}\\ \text{and } V_{\text{frustrum}} &= V_1 - V_2\\ &= \frac{\pi}{\sqrt{3}}(r_1^3 - r_2^3)\\ &= \frac{\pi}{\sqrt{3}}\left(\left(\frac{R+r}{2}\right)^3 - r^3\right).\end{aligned}$$

Thus we have re-expressed the volume of the frustrum in terms of our relevant variables R and r.

3 Maximal filling

In this section we will fix R and find what value of r gives us the maximum amount of filling. Recall that R is the radius of the large circle of pastry. We will find the maximum of the volume V with respect to r by differentiating and finding stationary points.

Differentiating with respect to r we get

$$\frac{dV}{dr} = \frac{\pi}{\sqrt{3}}\left(\frac{3}{2}\left(\frac{R+r}{2}\right)^2 - 3r^2\right)$$

and we need this to be 0 for stationary points:

$$\begin{aligned}
\frac{\pi}{\sqrt{3}}\left(\frac{3}{2}\left(\frac{R+r}{2}\right)^2 - 3r^2\right) &= 0 \\
\Rightarrow \quad \frac{3}{8}(R^2+2Rr+r^2) - 3r^2 &= 0 \\
\Rightarrow \quad 3R^2+6Rr+3r^2-24r^2 &= 0 \\
\Rightarrow \quad 21r^2-6Rr-3R^2 &= 0 \\
r &= \frac{6R \pm \sqrt{36R^2+4\cdot 21\cdot 3R^2}}{42} \\
&= \frac{6\pm\sqrt{288}}{42}R \\
&= \frac{1\pm 2\sqrt{2}}{7}R
\end{aligned}$$

We discard the negative solution as we need a positive value of r so we get

$$r = \frac{1+2\sqrt{2}}{7}R$$

We check that this is a maximum by differentiating again:

$$\begin{aligned}
\frac{d^2V}{dr^2} &= \frac{\pi}{\sqrt{3}}\left(\frac{3}{2}\cdot\frac{2}{2}\left(\frac{R+r}{2}\right) - 6r\right) \\
&= \frac{\pi}{\sqrt{3}}\left(\frac{3R+3r-24r}{4}\right) \\
&= \frac{3\pi}{\sqrt{3}}\left(\frac{R-7r}{4}\right) \\
&= \frac{\sqrt{3}\pi}{4}(R-7r)
\end{aligned}$$

so when $r = \dfrac{1+2\sqrt{2}}{7}R$ we have

$$\begin{aligned}
\frac{d^2V}{dr^2} &= -\frac{\sqrt{3}\pi}{4}R\cdot 2\sqrt{2} \\
&= -\sqrt{\frac{3}{2}}\pi R \\
&< 0
\end{aligned}$$

so it is indeed a maximum.

The idea is that as r gets bigger relative to R, the average radius of the pie gets bigger but the depth gets smaller (as R is fixed), so there is a trade-off, with the maximum volume occurring when

$$\begin{aligned} r &= \frac{1+2\sqrt{2}}{7}R \\ &\simeq 0.54R \end{aligned}$$

4 On mince to pie balance

In this section we examine the ratio of filling to pie. The volume of pastry comes from the area of two circles multiplied by the thickness of the pastry, which we will call t. So the volume of pastry is:

$$\begin{aligned} P &= (\pi R^2 + \pi r^2)t \\ &= \pi(R^2 + r^2)t. \end{aligned}$$

The volume of mince is

$$M = \frac{\pi}{\sqrt{3}}\left(\left(\frac{R+r}{2}\right)^3 - r^3\right).$$

So the ratio of mince to pie is

$$\begin{aligned} \frac{M}{P} &= \frac{\frac{\pi}{\sqrt{3}}\left(\left(\frac{R+r}{2}\right)^3 - r^3\right)}{\pi(R^2+r^2)t} \\ &= \frac{\left(\frac{R+r}{2}\right)^3 - r^3}{\sqrt{3}(R^2+r^2)t} \end{aligned}$$

- $\frac{M}{P} < 1$ means more pastry
- $\frac{M}{P} = 1$ means balance
- $\frac{M}{P} > 1$ means more mince

Note that the π's cancel out in the fraction; I wish the pies would cancel out in my stomach. Here are some examples.

Example 1

I have three sizes of pie mould. If I roll out my pastry to 0.5 cm thick, what pie balance will I get for my three moulds? Here are the approximate measurements of my three moulds, and the resulting mince:pie balance.

R/cm	r/cm	t/cm	$\frac{M}{P}$
1.1	2.5	0.5	0.7
1.5	3.5	0.5	1.0
2.0	5.0	0.5	1.4

Note that t is the cooked thickness of the pastry. It rises in the oven so needs to be rolled thinner than t to make it t afer baking.

Example 2

To make all three pies balanced, how thick should I roll my pastry?

R/cm	r/cm	t/cm	$\frac{M}{P}$
1.1	2.5	0.35	1
1.5	3.5	0.50	1
2.0	5.0	0.70	1

Note that I have only included the extra 0's after the decimal point to make the table line up.

5 Reality

Of course not all mince pies are perfectly frustal, nor do all moulds have 30° angles. My actual mince pie tray is much shallower than my individual moulds; I just used $\theta = 30°$ to make the calculations easier.

Also I don't usually fill my pies with mince up to the brim. Shop bought pies are more likely to be filled to the brim with a lid that completely closes off the top, rather than r being the radius of the base. I find that closing the pie tends to make my pies overflow, even if I make holes in the lid. Actually I usually put stars on top instead of circles, so that there's more leeway for the mince puffing up, and to distract from the wonkiness of my pies.

In any case I am of course not actually suggesting using maths to make perfect mince pies. My real formula for the "perfect mince pie" is: make it yourself! It's totally worth it.

The other way to have perfect mince pies is to have the ones made by my friend Julie. Julie's mince pies are perfect.