

Mathematics & Lego: the untold story

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When I tell people I'm a pure mathematician, they invariably say something like "Oh, so I won't understand what you do."

Well in the next 15 minutes I'm going to let out my best kept secret: my research is actually just like playing around with lego.

Lego: the universal toy. Enjoyed by people of all ages all over the place. The idea is simple and brilliant. Start with some basic blocks that can be joined together. Add creativity, imagination and a bit of ingenuity. Build *anything*.

Mathematics is *exactly the same*. We start with some basic building blocks and ways of joining them together. And then we use creativity, and, yes, imagination and certainly ingenuity, and try to build *anything*.

Of course, just like with lego, things start getting complicated. You get fed up with the limited range of possibilities when you've only got basic lego bricks. You might want to build a house - so you need door pieces, maybe window pieces. *Then* you can build all sorts of houses...until you start to long for something *more* complicated...

There are then all sorts of other pieces we can buy. There are in fact whole series of 'themed' pieces to cater for what kind of thing you want to build. There's castle lego, spaceship lego, star wars lego and so on. And there are lego fanatics who become particularly obsessed with one branch of lego, because they *just* want to build castles, or spaceships, or whatever. And there are some people who just want to use basic bricks and build huge and extraordinary sculptures from scratch.

Maths is the same. We start wanting to build more and more complicated mathematical things. So we might use more complicated building blocks, or we might come up with more and more ingenious ways of sticking the blocks together.

The idea of Category Theory

I work in Category Theory. I study mathematical structures - how they are built in general. Why do we need to do this? Well if we only want to build something simple then we probably can just follow our nose and use our instinct. But if we want to build complicated things we might need to understand properly the principles behind building. Personally, I could take some lego and build an ordinary little house. But could I build...*that?* I would need to understand the principles of building huge things in order to do that.

Now we could go round this room and see what's the most complicated thing we think we could build from lego off the top of our heads. We'd probably have wildly varying capabilities. Perhaps there's a lego fanatic in here who could build something like...*this*...?

Well, it is possible to help other people build brilliant lego models. If I go out to the shops I can buy a lego set which will explain to me the principles of building, or how to build a firestation, or a garage or something. If I only want to build something simple, I don't need the instructions. But if I want to build something complex then I do.

Maths is the same. A lot of maths can be done without really thinking about *how structures are built*. That's why Category Theory, only came into existence about fifty years ago. That's when people realised that things were getting complicated, and we needed to work out the principles of building in order to *help other mathematicians build good mathematical structures*. That's what Category Theory is trying to do.

How mathematical structures are built

So how *do* we build mathematical structures? Or, put another way, how does lego work?? Imagine explaining to a lego virgin how lego works. Perhaps there's someone here who doesn't know what lego is??

We would explain it slowly, bit by bit.

At the most basic level we have basic bricks. I don't know about you but **this** is what I consider to be the generic lego brick. What's important about a lego brick? The key thing is that they *can be stuck together*. How? Well, for a start, like **this - just on top**. Well, our lego novice might then interrupt and say "hang on, hang on, let me try this first" and they will start doing **this**. Ok ok I understand that now - you can't do much can you? No, but also, you can stick them like **this** - ok let me try that...ah, now I can build a wall. But now try **this** ...AHA now this IS getting exciting. I can go round corners. Perhaps I can build a house? But how will I get doors and windows?? Well that's ok, because actually there are other lego pieces like door pieces and window pieces, so you CAN build a house...

So you see, we can gradually introduce more and more things to our lego system to build more and more complicated things. Fundamentally, we start with *building blocks*, and ways *of sticking them together*. We can then introduce *more ways of sticking them together*, or *more kinds of building block*.

Different philosophies

This is where lego fanatics - and mathematicians, go off in different philosophical paths. Think about it. Would you be more interested in using basic lego bricks to build fantastic big sculptures? Or would you be interested in buying all the complicatedly designed little pieces to build machines or working robots or trainsets?

My work

Personally, I'm a purist. I am really amazed by using just basic bricks to build the Eiffel Tower or something. I think this is brilliant and clever and beautiful. I'm not particularly interested in something like **this** that uses a whole load of bits that seem to have nothing to do with lego at all! Is that clever? Is that beautiful? OK, I suppose I should be open minded and say...it's *different*...but it just doesn't get me excited.

And I'm the same with maths. I'm a purist, a *pure* mathematician. I don't like complicated gadgets and machinery. I like taking basic bricks - mathematical bricks - and studying the possibilities...

Studying the possibilities of a system.

A mathematical system is like a lego set. You have certain types of pieces - perhaps just basic pieces, perhaps 'castle pieces' or 'spaceship pieces', that can be stuck together in certain ways.

A mathematical structure is what you build with it.

So really, mathematicians are just sitting around playing with their lego. Well, perhaps I should just speak for myself. I sit around and play with my lego.

High dimensions

So far so good. We like lego. It's fun. It's simple.

Now imagine that you've woken up in a 4-dimensional world. And I don't mean time as the fourth dimension. I mean four space dimensions. Weird and exciting. Now suppose they had no lego. Disaster! Calamity! Right, you think - *I* will bring lego to the 4-dimensional world!

This is what my research is about. Except in the mathematical world there aren't just 4 dimensions either - we can have as many dimensions as we want, four hundred, four million, four zillion, four squillion...but so far we've only got 3-dimensional lego. So there are rather a lot of things out there that we can't build because our lego is too feeble.

Imagine for a moment a lego set with only pieces like **this**. You would not be able to do anything very interesting, and there would be rather a lot of things that you wouldn't be able to build - like houses, castles and anything really. It's easy to understand, but frankly rather feeble.

So we as mathematicians are trying to sort out the higher dimensions.

But what on earth does this mean? Does my analogy still work? If you found yourself in a 4-d world, how would you go about making 4-dimensional lego?

Well, it's hard, especially as you're new to this extra dimension so don't have a full understanding of how it all works. You just know that the possibilities in four dimensions are suddenly incredible. We can try a few things.

Generalise

You might sit down and think very hard about how lego works in 3-dimensions, to see if we can do the same sort of thing in 4-dims. In my work I do just this - I study very carefully the

structures in low dimensions that are understood, to try to extend the principles into higher dimensions.

Look for evidence

You might also try to familiarise yourself a bit more with the 4-d world you're in. What's a 4-d house? What's a 4-d car? What's a 4-d bottle? If we can understand the workings of the 4-d world a bit better, we'll be a bit closer to being able to make 4-d lego. In my work, I do just this - there *are* higher-dimensional mathematical structures 'out there' so I study those carefully to try and understand the *principles* at work.

Come up with a prototype and try it out

Once you think you have a reasonable understanding of these things you might be able to come up with a prototype, a first guess at what 4-d lego could be. And then you test it in various ways. First of all you might just see if it works - do the pieces fit together properly? Do they fit tightly so that the structure will be secure?

Once you're happy that the pieces fit together you could start trying to build things. You might wander round the 4-d world trying to build the things that you see. You might be able to build some things but not others. So you go back to the workshop to see if you can modify your lego so that you can build those other things...and then you try again...

And yes, this is what I do in my work as well. We have a prototype - and really it's still only the *basic bricks* we're trying to work out. We have to see if the bricks will fit together properly - which is what we call coherence theory. And we also go back and wander round the mathematical world, looking at the higher-dimensional structures and seeing if we can build them with our basic bricks.

The state of the art

So how far have we got? Well imagine that there are rival factions all trying to make what they think is the best 4-dimensional lego. They're all very obsessed, and they think of course that their lego is the best. But they hardly even *look* at what anyone else is doing. They just glance and say either "that's rubbish" or else accuse them of stealing their design.

But then some young idealist comes along and thinks: oh, wouldn't it be so much more productive if we all worked together, and we can then benefit from everyone else's wisdom, and work in harmony rather than fight against one another, united we stand, divided we fall, etc etc

Well in the world of mathematical lego, that young idealist is me.

The end

And this young idealist is now going to let you into another secret: I'm actually not very good at lego.

Last night someone asked me about this seminar and I said: in summary, my work is just like playing around with lego. She looked at me completely incredulously and said “Yes but maths is hard - lego is easy” And I thought - *is it?* If you went and looked round Legoland, could you build things like that? Well *I* certainly couldn't...

...so I guess maths isn't really like playing around with lego after all. Because I can sort of *do* maths. To be honest, I'm just not that great at lego. Which just shows - all analogies start getting tenuous if you take them too far, and I think I've stretched this one quite far enough.